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Should I Stay or Should I Go? Optimal Decision Making in Penalty Kicks. Comment on Bar-Eli et al.

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Abstract[1]

Bar-Eli et al. (2007, hereafter BE) conclude that goalkeepers in football suffer from an action bias that preserves them from optimally playing in penalty kicks. Particularly, they are thought to jump too often to the sides and to stay too seldom in the centre. This bias is explained by a norm that goalkeepers should jump to the sides in order to minimize their mental costs when failing to stop the ball. Because in this case at least they have done something instead of just watching how the kicker converts the penalty. We show that these conclusions are wrong using the data set of BE, the data set of the criticized Chiappori et al. (2002), an own data set consisting of 1043 penalty kicks from the German Bundesliga and data of the likewise criticized Palacios-Huerta (2003). The crucial mistake of BE consists of modeling the strategic interaction between the players as if it was a parametric decision. A game theoretic analysis that takes into account the middle as an option shows that goalkeepers on average behave astoundingly close to their optimal choices. Therefore, the action bias BE identify in an enquiry of goalkeepers actually is a rule of thumb that helps the goalies to maximise the chance of stopping the penalty kick. Finally, it is shown that players are not only rational in choosing their sides in a sole penalty kick but also in a series of kicks.

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1 Introduction

Studies of sports have become popular to analyse interaction in real-world settings. Particularly, serving in tennis (Hsu et al., 2007; Walker and Wooders, 2001) and kicks in football (Chiappori et al., 2002, hereafter CH; Moschini, 2004; Palacios-Huertas, 2003, hereafter PH), where an actor can choose between two actions (shoot or serve to the left or to the right), have led to the conclusion that the behaviour of professional players can be explained by game theory.

Bar-Eli et al. (2007, hereafter BE) expand this analysis to penalty kicks that are modeled so that both players can choose between three actions (left, centre, right). They find that in this more complex situation goalkeepers are far from behaving optimally. Rather – according to BE – they decide to jump to the sides too often, instead of staying in the middle where their chance of stopping the ball is highest. BE explain this fact by a normative action bias. It drives the goalkeeper not to stay in the middle, because in case of a failure[2] at least he has shown some action and tried his best instead of passively watching the ball pass. BE corroborate this result with an enquiry of the attitudes of professional goalkeepers towards actions during penalty kicks.

We claim that these findings are wrong. Especially, this is the case because BE model the strategic decision of the penalty kick as if it was a parametric one. Consequently, their explanation for the goalkeepers failure is incorrect, too. Using BE's data and three additional data sets, we show that a correct analysis of penalty kicks with three strategies leads to the finding that goalkeepers still behave astonishingly rational. In addition, this is not only true in a single but also for a series of kicks.

The comment is organized as follows: First, optimal behaviour in a penalty kick with three strategies will be analysed theoretically for a single and a series of kicks. Secondly, these findings are tested empirically in section 3.

2 Strategic interaction in penalty kicks

The theory used here is standard game theory available from textbooks (see e.g. Dixith and Skeath, 2004 for illustrations with sports). We renounce a formal deduction of the problem here and instead give a verbal explanation starting from BE's work. The formal application of the model to penalty kicks is demonstrated in Berger and Hammer (2007, hereafter BH), Chiappori et al. (2000), CH, Palacios-Huertas (2002) and PH. We do neither present the game itself because a description is also available in the latter texts and BE.

2.1 Strategies in a single kick

To show that BE's analysis is problematic, we start from the assumption that there really is an effective norm for the goalkeeper to jump. Using BE's words then it follows that "the kicker, in turn, has an incentive to surprise the goalkeeper" (BE, 4) by kicking to the centre. Now, of course the goalkeeper will overturn his decision and *not* jump precisely *because* there is a norm to do it. The kicker knowing that, in turn, will overthrow his decision and again choose one of the sides, so that the goalkeeper also reverses his decision, et cetera.

[2] This is the case in about four out of five kicks.

This infinite regress is typical for a zero-sum game of which penalty kicking is an example. An intuitive solution of this circle of expectations is not obvious.

Nevertheless, the solution is one of the first findings of game theory, the so called Minimax Theorem (von Neumann, 1928).[3] For the case of penalty kicks it states that both players should equalize their expected winning rates for all of their possible actions. If the game is modeled with only the options left and right, then the goalie should act so as to equalize his chance of stopping the ball on both sides *and* the kicker should do the same to equalize his scoring rate on both sides. Because these two rates obviously are interdependent, the players do *not* orient their action on their *own* expected payoffs *but* on the payoffs of their *opponents*. Formally this shows up in the fact that the equation for the goalkeeper's choosing probabilities can be calculated by taking into account the *kicker's* payoff (see e.g. BH, Appendix). If a player deviates from this behaviour, he would become exploitable. His opponent then would *always* choose the side where his winning rate was lower (and therefore the opponent's winning rate higher). A good illustration for this relationship is the case where the goalkeeper is handicapped and has only one arm. His stopping rate on the armless side will then obviously be lower than on the other side. The kicker will then always shoot to the armless side where his scoring rate is higher. This means that the goalkeeper will choose to jump more often to the armless side (or more realistic: will shift on the goal line into the armless direction), until the scoring rates of the kicker and therefore his stopping rates are equal on both sides.

This analysis not only holds for the model with two options (as BE implicitly assume) but can be extended to virtually every area of the goal (e.g. to the nine fields of the goal BE used for their data collection, cf. BE appendix A). For optimal behaviour in the 3 x 3 game which takes the centre into account, this means for the goalkeeper that the stopping probabilities for all three actions must be the same, and *not* "to choose the direction where his probability of stopping the ball is maximal" (BE, 8). From the high winning rates of the goalkeeper in the centre, it follows that the kicker's winning rate is low in this direction. So indeed, the goalkeeper should *not* stay there but in most cases jump to one of the two sides where the kicker's scoring rates are high. This follows not from an action bias or a norm but simply from optimising behaviour. Moreover, this analysis also leads to other rather counterintuitive predictions, e.g. that in optimal equilibrium behaviour the goalkeeper should jump even more often to one of the sides than the kicker kicks there (for a formal derivation see CH). This prediction theoretically hold for every single kick, but they can only be tested empirically with a sample of penalty kicks where the estimation of the expected payoffs is possible.

2.2 Strategies in a series of kicks

BE give a hint that a goalkeeper could try to guess the kicking direction of a player by knowing the history of his previous penalty kicks (BE, 4). Maybe a kicker in the past has favoured one side and might favour the same side for his actual kick. But of course the same argument as for the single kick applies here. Just because the goalkeeper thinks that the kicker will choose his favoured side again, the kicker will change to the other side.

[3] The Minimax solution is a special case of the Nash equilibrium (Nash, 1950) which is more general and holds for all lifelike strategic situations. BE (p. 3) use the term "mixed-strategy Nash equilibrium (MSNE)" for it.

Expecting this, the goalkeeper will also reverse his decision, and so on. Again, the solution for this infinite regress is an equilibrium where no player has an incentive to deviate from his decision as long as the other player does not deviate neither. This implies that the goalkeeper's actual decision must be incalculable from his previous actions. Therefore, the goalie should decide as if he chose a jumping direction randomly. This results in serial independency between two consecutive actions of one player.[4]

3 Empirical tests

The empirical work starts with the presentation of the evidence from laboratory experiments. It continues with the presentation of the data sets and the results for single and a series of kicks, respectively.

3.1 Evidence from laboratory experiments

There is a large body of evidence from laboratory experiments concerning the behaviour of humans in interactions with mixed strategy equilibriums. Detailed summaries can be found in BH, CH, and PH. Not all of the evidence applies exactly to the examined situation. But it can be stated that untrained human beings tend to fail in optimising their behaviour in a series of decisions. An experiment of Palacios-Huerta and Volij (2008) implies that this might indeed be due to the lack of experience (but cf. also Levitt et al. 2007). Nevertheless, in single decisions they act fairly close to optimal behaviour in situations with a single decision.

3.2 Data sets

We use four data sets to test our hypothesis. The data sets of CH, PH and BE are known and will not be presented. In the available form these three data sets are only suitable for testing the single kick forecasts. The fourth data set is available to us on the individual level of the players and can be used to test predictions about a series of kicks. It consists of data of all 1043 penalty kicks that occurred during the seasons 1993/94 to 2003/04 in the first league of Germany (Bundesliga). It contains all information (and some more) necessary for this test.[5] The data was collected by four professional observers coding the same game independently and simultaneously. This data collection takes place routinely in order to sell up-to-date data sets to the observed teams and their opponents. So the data can be expected to be most objective.

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4 While BE's idea that the goalkeeper should prefer the centre in penalty kicks might have astounded amateurs of the game, this is not the case with this second hypothesis. On the contrary, there is always a discussion among football amateurs that a clever goalkeeper could calculate the kicking direction of the kicker. E.g., during the last world tournament there was the commonly hold belief, that the German goalkeeper Lehmann learned the kicking direction of his Argentine opponents from a note given to him before the penalty shootout of the quarter-final. Indeed, he stopped three balls and Germany won. But – according to game theoretical predictions – it turned out that Lehmann did not know the kicking directions from this note.

5 Detailed information about the data set can be found in BH.

Table 1 to 3 show the descriptive parameters of three data sets.[6] It can be seen, that there are only slight differences between the crucial values. This is a hint that the game is played about the same in different leagues, which could be due to equilibrium play.

Table 1: Data Set of Chiappori et al. 2002: joint distribution of the strategies of kicker and goalkeeper (scored goals in parentheses).

		Goalkeeper							
		left		centre		right			
	left	75	(33)	4	(4)	95	(85)	174	37.91%
Kicker	centre	28	(25)	3	(0)	48	(39)	79	17.21%
	right	85	(80)	4	(4)	117	(74)	206	44.88%
		188	46.96%	11	2.40%	260	56.46%	459	100%

Data sources: CH table 3 and 4. All directions are noted from the goalkeeper's view. Shots involving left-footed kickers have been noted reversed, so that shooting right corresponds to the “natural” side for all kickers.

Table 2: Data Set of Bar-Eli et al. 2007: joint distribution of the strategies of kicker and goalkeeper (scored goals in parentheses).

		Goalkeeper							
		left		centre		right			
	left	54	(38)	1	(1)	37	(37)	92	33.17%
Kicker	centre	41	(37)	10	(4)	31	(30)	82	28.67%
	right	46	(46)	7	(7)	59	(44)	112	39.16%
		141	49.30%	18	6.29%	127	44.41%	286	100%

Data sources: BE table 1 and 2. All directions are noted from the goalkeeper's view.

Table 3: Data set of Berger and Hammer 2007: joint distribution of the strategies of kicker and goalkeeper (scored goals in parentheses).

		Goalkeeper							
		left		centre		right			
	left	202	(106)	6	(5)	225	(216)	433	41.51%
Kicker	centre	62	(46)	3	(1)	86	(55)	151	14.48%
	right	220	(202)	8	(8)	231	(149)	459	44.01%
		484	46.40%	17	1.63%	543	52.06%	1043	100%

All directions are noted from the goalkeeper's view. Shots involving left-footed kickers have been noted reversed, so that shooting right corresponds to the “natural” side for all kickers.

[6] The data set of PH is not used here, because it can not be reconstructed from the corresponding article due to rounding procedures.

3.3 Results for single kicks

Table 4 reports the actual and expected frequencies for all three data sets if the game is modeled only with the strategies "left" and "right". The estimation of the expected optimal frequencies of the goalkeepers is done by aggregating the corresponding data from Table 1 to 3.[7] It reports the optimal distribution of "left" and "right" for them, given the actual play of the kickers.[8] The corresponding estimation for the data of BE is surely slightly biased for two reasons. The first one is the fact that 18 kicks that missed the goal or went to the goalposts or the crossbar and that were excluded from the data set by BE (p. 6) in order to estimate the goalie's optimal behaviour. But – as it is shown above – the optimal behavior of the goalies is determined by the expected payoffs of the kickers, so that these 18 kicks should have been taken into account.[9] The second reason is the missing correction for left-footed kickers that also influences the goalkeepers decision (see footnote 6).

Table 4: Expected and actual frequencies of strategies in percent for goalkeepers *without* taking the centre into account (2 x 2 game) for all four data sets.

	CH		PH		BE		BH	
	left	right	left	right	left	right	left	right
expected	29.44	70.56	41.99	58.01	41.59	58.41	47.16	52.84
actual	41.96	58.04	42.31	57.69	52.61	47.39	47.17	52.83

Data sources: CH table 3 and 4, PH, 402, BE table 1 and 2, and own calculations. All directions are noted from the goalkeeper's view. The strategy "right" denotes the shot respectively jump to the natural side of the kicker, with the exception of the data set of BE.

It can be seen that the goalies do fairly well in optimising their choices. In particular the goalkeepers in the Bundesliga (BH) and those from the data set of PH make near perfect decisions. Yet, the goalies in the data set of CH and those of BE could ameliorate their play. However, the difference in the data of BE might also stem from the missing correction for left-footed kickers and the exclusion of 18 kicks from the data by BE (cf. above).

7 Following CH, kicks to the centre then are counted to the so called "natural" side of the kicker. This is the left side for right footed kickers and vice versa. The very few (cf. Table 1 to 3) cases where the goalkeepers stayed in the middle are excluded from this estimation. The exact justification for this procedure can be found in PH (see also CH and BH). For the data of BE this procedure is not followed because the required information is not known. Therefore, we expect the findings to be slightly biased for the data set of BE.

8 These calculations can be done easily by hand for the 2 x 2 case (see appendix of BE for the equation). Yet, calculations become rather complicated (though not complex) already in the 3 x 3 case. We therefore used the software "Gambit" to estimate the optimal behaviour of the players. "Gambit" is free software and can be downloaded from http://econweb.tamu.edu/gambit.

9 Besides, this exclusion is the reason for the unusually high scoring rate of the kickers (85.3%) in the data of BE.

Table 5: Expected and actual frequencies of strategies in percent for goalkeepers taking the centre into account (3 x 3 game) for three data sets.

	CH			BE			BH		
	left	centre	right	left	centre	right	left	centre	right
expected	31,25	9,22	59,53	41,10	11,01	47,89	44,49	0,0	55,51
actual	40,96	2,40	56,46	49,30	6,30	44,40	46,40	1,63	55,06

Data sources: CH table 3 and 4, BE table 1 and 2, and own calculations. All directions are noted from the goalkeeper's view. The strategy "right" denotes the shot respectively jump to the natural side of the kicker, without the exception of the data set of BE.

According to BE this rather good corroboration of game theory is due to the exclusion of the middle as an option for both players. Therefore taking into account the centre as an option should lead to an overall deterioration of the game theoretic predictions. The calculations follow the same logic as the one for the 2 x 2.

Table 5 reports the corresponding results.[10] Indeed the results change a little bit but not in the direction BE expect. Firstly, there is no hint that the centre is the crucial option for the goalkeepers, not to speak of the importance BE give it in their article. In fact, optimally behaving goalkeepers from their data set should stay in the middle in about 11% of all kicks instead of the 6.3% when they really do.[11] About the same holds for the goalies in the data set of CH who stay in the middle only in 2.4%. They could improve their performance by doing this more often (9.2%).

This could lead to the conclusion that there indeed is some sort of action bias driving the goalkeepers to jump more often to the sides than they should (even if it would be much weaker than BE expect it to be). But the results for the Bundesliga falsify this suggestion. There, the goalkeeper should *never* stay in the middle given the distribution of kicks. The sides completely dominate the centre. Again this somehow paradoxical result reflects the fact that in a mixed strategy equilibrium players orient their behaviour on their opponents expected payoffs instead of their own.[12] Particularly the goalkeepers in the Bundesliga anticipate this nearly perfect by staying in the middle only in 1.6% of all kicks.

Now the attitudes of the goalkeepers towards choosing the centre that found BE are easily explained. They do not reflect an action bias but a useful rule of thumb for goalkeepers trying to stop a penalty kick.[13] A rule of thumb (not a norm) that reads like

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[10] For the data set of PH this analysis cannot be done with the data available to us.

[11] This still rather high rate of "centre" in the optimal mixture might also be affected by the mistakenly excluded data.

[12] One could expect that the probabilities for the sides should stay the same once the centre is no option in optimal play. This is not the case because collapsing the 3 x 3 game into a 2 x 2 game leads to the exclusion of the few shots where the goalkeeper stayed in the middle (cf. above). This gives the right side of the kicker a little more and the left side of the goalie a little less weight.

[13] Besides, also the other expressed attitudes of the goalkeepers reflect their correct anticipation of the kickers. 16 out of 25 goalkeepers preferred the right side to the left. This is predicted by game theoretic analysis and empirically corroborated by the above data (cf. tables 1 to 3; BH, p. 9f.; CH, p. 1142ff.). Having asked BE questions (appendix C) more accurately with the supplement "..., given the kicker is right footed" (which corresponds to the majority of kickers, cf. BH table 1; PH table 1) probably would have led to an even higher rate of suitable answers.

"never stay in the middle, as long as you do not get a sure hint from a bad kicker that he will kick to the centre" would fit the data well.[14]

3.4 Results for series of kicks

In order to do the test for serial independency of the jumping decision we need information about the actions of individual goalkeepers that were involved in several penalty kicks. The Bundesliga data set offers this information for 13 goalkeepers that were engaged in more than twenty penalty kicks. The estimation can only be done for the sides because there are far too few cases where the individual goalkeeper chose the centre to make any statistical statement about it (cf. table 3). To test for serial independency of decisions the actual number of "runs" is compared to the expected number. A run is a series of the same decision.[15] For instance, the series LLRL has three runs. The higher the number of runs the more often the player has changed his decision. Table 6 shows the corresponding results for the goalkeepers.

Table 6: "runs"-test for serial independency of decisions

Goalkeeper	n	n_L^i	n_R^i	$runs_{exp}$	$runs_{act}$	z	p-value
Jörg Butt	23	7	16	11	9	-0.63	0.53
Richard Golz	43	25	18	22	27	1.77	0.08*
Dirk Heinen	24	12	12	13	11	-0.63	0.53
Oliver Kahn	33	17	16	17	14	-1.06	0.29
Gabor Kiraly	24	7	17	11	10	-0.21	0.83
Stefan Klos	21	10	11	11	10	-0.44	0.66
Georg Koch	27	9	19	12	10	-0.83	0.40
Jens Lehmann	33	15	18	17	21	1.48	0.14
Martin Pieckenhagen	28	12	16	15	13	-0.48	0.63
Oliver Reck	47	23	24	24	23	-0.29	0.77
Claus Reitmaier	40	29	11	17	16	-0.18	0.86
Frank Rost	36	18	18	19	22	1.18	0.24
Jörg Schmadtke	21	11	10	11	11	0.01	0.99

Abbreviations: n_L^i and n_R^i denote the number of jumps of the goalkeeper i taken to the left and right side, respectively. $runs_{exp}$ denotes the number of jumps expected under serial independency, and $runs_{act}$ the number of actual runs. There is a continuity correction for small numbers of cases. * denotes players where H_1 is refuted with 90% or more.

Again, the centre is excluded from the data set for this test because of too few cases. It can be seen that only one goalkeeper (Golz) fails to decide randomly in his series. Another (Lehmann) is on the brim of non-randomness. Both of them changed their decisions too often (too many runs). This behaviour is known from similar situations with non-experienced actors in the laboratory. But taking into account that each player corresponds to one experiment with a significance level of 90%, we expect 1.3 false-negative tests

14 BE (appendix B) find that in about 10% of the kicks the players may not have acted simultaneously. In some of these cases the goalkeepers could have decided to stay in the middle *after* having detected the kicking direction.

15 We do not go into details of the application of the "runs"-test to the data. These can be found e.g. in BH; Swed and Eisenhart, 1943; Walker and Wooders, 2001.

among 13 goalies. So, in summary as it is known from the previous evidence, professional players seem to be able to make random decisions in series.

4 Conclusion

In their article BE make a case against game (or in their interpretation: rational choice) theory by stating that in the highly competitive situation of penalty kicks goalkeepers are driven by a norm that is directed against their optimal behaviour. BE claim that this bias can be detected when the penalty kicks are modeled with three strategies for both players instead of two as it has been done before. Our estimations refute these findings. It is shown that on average goalkeepers engaged in penalty kicks do not suffer from an action bias, but instead behave astoundingly close to the optimal mixture of strategies. We show that this is also true for those individual goalies that were involved in several penalty kicks. Consequently, the attitudes of the goalkeepers towards penalty kicks are rule-of-thumb guidelines for optimal play and not the expression of an action bias to reduce mental costs after failing. In addition, the finding that goalkeepers behave rational is also confirmed for a series of kicks. So, in summary, game theoretical predictions are corroborated in a zero-sum game with huge amounts of money and social status on stake, where indeed the existence of an effective norm directed against optimal behaviour would have been a major refutal of the paradigm.

However, it is still not known what kind of underlying rationality is at work here. The rationality of the players could be seen rather as heuristics to explain emerging phenomena on aggregate level (cf. e.g. Smith, 2003). Yet, because professional players perform better than non-experienced actors, rationality here also seems to be the result of a subconscious individual adaption to a certain environment (cf. e.g. Raab and Johnson, 2006). If this were true, data sets that cover the complete environment (say: all penalty kicks in a certain time span and league) as the ones of CH and BH should be more informative than those with unknown and/or selected kicks as the ones of PH and BE. In any case the crucial experiment to be done would be to observe untrained players in a corresponding real world setting.

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